A derivation of Poynting's Theorem in relativistic electromagnetism

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Background

This investigation will use the metric tensor

$$g = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

It gives the isomorphism between contravariant vectors and covariant dual-vectors. Under this convention, the contravariant field tensor in Cartesian coordinates is

$$F = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

and Maxwell's equations read

$$\partial_\mu F^{\mu\nu} = \mu_o J^\nu$$

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0$$

In another notation that I will use, the first of these equations reads

$$\square \cdot F = \mu_o J$$

where the dot product includes the metric tensor as usual, and the 4-gradient is the contravariant derivative operator:

$$\square = \begin{pmatrix} \frac{1}{c}\partial_t & -\nabla \end{pmatrix}$$

Define the stress-energy tensor as follows:

$$T = \frac{1}{\mu_o}\left(F^2 - \frac{1}{4}\mathrm{Tr}(F^2)g^{-1}\right)$$

which is the same as

$$T^{\mu\nu} = \frac{1}{\mu_o}\left(F^{\mu\rho}F_\rho^\nu - \frac{1}{4}F^{\rho\lambda}F_{\lambda\rho}g^{\mu\nu}\right) = \frac{1}{\mu_o}\left(F^{\mu\rho}F_\rho^\nu + \frac{1}{4}F^{\lambda\rho}F_{\lambda\rho}g^{\mu\nu}\right)$$

This has the form

$$T = \begin{bmatrix} u & S/c \\ S/c & -\Sigma \end{bmatrix}$$

Where u, S, and Σ are the usual expressions for field energy density, Poynting vector, and 3D stress tensor, respectively.

Derivation of the Theorem

The desired result is found by computing the divergence of this tensor:

$$\Box \cdot T = \frac{1}{\mu_o}\left(\partial_\mu (F^{\mu\rho} F_\rho^\nu) + \frac{1}{4}\partial_\mu (F^{\lambda\rho} F_{\lambda\rho}) g^{\mu\nu}\right)$$

$$= \frac{1}{\mu_o}\left(\partial_\mu (F^{\mu\rho}) F_\rho^\nu + F^{\mu\rho}\partial_\mu \left(F_\rho^\nu\right) + \frac{1}{2}\partial_\mu (F^{\lambda\rho}) F_{\lambda\rho} g^{\mu\nu}\right)$$

$$= \frac{1}{\mu_o}\left(\partial_\mu (F^{\mu\rho}) F_\rho^\nu + F^{\mu\rho}\partial_\mu \left(F_\rho^\nu\right) + \frac{1}{2}\partial^\nu (F^{\lambda\rho}) F_{\lambda\rho}\right)$$

From the second of Maxwell's equations,

$$\partial^\nu (F^{\lambda\rho}) = -\partial^\lambda F^{\rho\nu} - \partial^\rho F^{\nu\lambda}$$

So

$$\frac{1}{2}\partial^\nu (F^{\lambda\rho}) F_{\lambda\rho} = -\frac{1}{2}\left(\partial^\lambda F^{\rho\nu} + \partial^\rho F^{\nu\lambda}\right) F_{\lambda\rho}$$

and we can exchange the dummy indices λ and ρ in the second term, to get

$$\frac{1}{2}\partial^\nu (F^{\lambda\rho}) F_{\lambda\rho} = -\frac{1}{2}\left(\partial^\lambda (F^{\rho\nu}) F_{\lambda\rho} + \partial^\lambda (F^{\nu\rho}) F_{\rho\lambda}\right)$$

Now, because of the antisymmetry of the field tensor, reversing the indices of any F introduces a minus sign, and doing so for two F's being multiplied together leaves the result unchanged, so we have

$$\frac{1}{2}\partial^\nu (F^{\lambda\rho}) F_{\lambda\rho} = -\frac{1}{2}\left(\partial^\lambda (F^{\rho\nu}) F_{\lambda\rho} + \partial^\lambda (F^{\rho\nu}) F_{\lambda\rho}\right) = -\partial^\lambda (F^{\rho\nu}) F_{\lambda\rho}$$

All together, we now have

$$\square \cdot T = \frac{1}{\mu_o}\left(\partial_\mu (F^{\mu\rho})F_\rho^\nu + F^{\mu\rho}\partial_\mu\left(F_\rho^\nu\right) - \partial^\lambda (F^{\rho\nu})F_{\lambda\rho}\right)$$

In the last term, both λ and ρ are dummy indices. This means that we are free to raise them in one place while simultaneously lowering them in the other. This as a consequence of the trivial identity

$$gg^{-1} = I$$

I will also rename the dummy index λ as μ, in order to make the coming cancellation more obvious. Then the last term becomes

$$-\partial_\mu\left({F_\rho}^\nu\right)F^{\mu\rho}$$

which cancels exactly with the term before it. So

$$\square \cdot T = \frac{1}{\mu_o}\partial_\mu (F^{\mu\rho})F_\rho^\nu$$

From the first of Maxwell's equations, this becomes simply

$$\square \cdot T = J^\rho F_\rho^\nu = -{F^\nu}_\rho J^\rho$$

which is precisely the negative of the Lorentz force density on charged matter. This is Poynting's Theorem, giving both the energy flux and the momentum flux in one simple equation.

Discussion

The tensor *T* plays a very important role in the dynamics of any system. Poynting's Theorem turns out to be the electromagnetic version of a general Navier-Stokes type equation. There is a sense in which the expression for the 4-force

$$\frac{dp}{ds} = -\square \cdot T$$

is fundamental. Once one writes down a stress-energy tensor for some field constituent of nature and specifies field equations, then the Lorentz force of the interaction of that field with other constituents of nature is completely specified by this equation. If one takes conservation of energy and momentum to be the fundamental principles, then specifying *T* automatically specifies not only all the physical content of a field (its energy, momentum, and pressure), but also all of its dynamics.

The generality of the relativistic Navier-Stokes equation becomes even more impressive when one combines all the constituents of nature into a single stress-energy tensor, by taking the sum of the individual stress-energy tensors. Then the dynamical equation reads

$$\square \cdot T = 0$$

This is manifestly the expression for local conservation of both total energy and total momentum, but I also claim that this equation simultaneously specifies all of classical mechanics, in particular, all three of Newton's laws of motion. I will now discuss how this equation implies the three laws.

1. Inertial reference frames. We are assuming that dynamics are specified by the conservation law $\Box \cdot T = 0$. We want to consider arbitrary space-time coordinate transformations. If this equation holds in one frame of reference, then by the invariance of the zero-vector under transformation, it will automatically hold in any frame reached by a coordinate transformation under which the divergence of a tensor *transforms like a vector*. A straightforward investigation of the transformation properties of derivatives reveals that such coordinate transformations are precisely the linear transformations. Hence we call any coordinate frame reached by linear transformation from the first one an *inertial* frame. These transformations include all the usual isometric transformations, such as all rotations, translations, reflections, and Lorentz boosts, but they also include many non-metric-preserving transformations as well. $\Box \cdot T = 0$ will still hold, so long as one replaces g with its appropriately transformed version.

2. Force equals change in momentum. This is equivalent to identifying the 4-force with $\Box \cdot T$ of an individual constituent of nature. This is the usual "mathematical" content of the equation.

3. Action-Reaction. Suppose we write T as the sum of two parts, say mechanical and electromagnetic. Then

$$\Box \cdot T_{mech} + \Box \cdot T_{EM} = 0$$

and it becomes clear that these two local vectors, representing the 4-forces of each constituent on the other, must be of equal magnitude and opposite sign. Furthermore, this observation is an excellent illustration of the principle of no action at a distance. Charged particles do not exert a force on other charged particles. A charged particle exerts a force on the electromagnetic field, equal and opposite to the force the field exerts on the particle. The momentum is transferred via the local field.